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Structural Stability of Filippov Automata

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Abstract

We introduce Filippov automata which are based on Filippov's theory of dynamical systems governed by piecewise smooth vector fields. We develop some of the global and generic aspects of the dynamics of Filippov automata. In particular we establish a generic structural stability theorem for two dimensional Filippov automata, which is a natural generalization of Mauricio Peixoto's classic result [4].

1 Introduction

Consider a hybrid automaton with two locations whose continuous state space is the 2-sphere. The initial conditions for the first location are the upper hemisphere and for the second location the lower hemisphere. The locations have vector fields X_- and X_+, respectively, and each location has an outgoing edge whose enabling condition is the equator of the sphere. This gives an

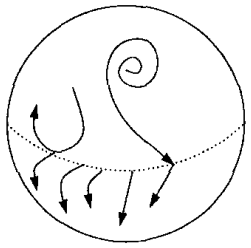

Figure 1: Hybrid orbits on a 2-sphere.

orbit portrait on the sphere. See Figure 1. What can it look like? How do perturbations affect it? How does it differ from the standard vector field case in which $X_+ = X_-$? These topics will be put in proper context and addressed in this paper.

Our work has precursors in an announcement by V.S. Kozlova [2] about structural stability for the case of planar Filippov systems, and also the work of Jorge Sotomayor and Jaume Llibre [3]. In Filippov [1] local conditions for structural stability are obtained, while our results are global.

2 Filippov Automata

A *Filippov automaton* is a dynamical system defined by the tuple

$$H = (I \times M, D, K, X^0).$$

$I \times M$ is the hybrid state space. I is a finite index set corresponding to the locations of H and M is a smooth, orientable, boundaryless, compact surface. $D : I \to \mathcal{X}^r(M)$ is map assigning a C^r vector field to each location, where $\mathcal{X}^r(M)$ is the set of C^r vector fields on M. $K \subset M$ is a fixed, smoothly embedded, finite 1-complex. The angle between edges of K meeting at a vertex is non-zero. That is, K has no cusps. The set of connected components of $M \setminus K$ is denoted $G = \{G_1, \ldots, G_k\}$. The connected components that abut across an edge of K are pairwise distinct. That is, K locally separates M. $X^0 : I \to G$ is a map that assigns initial conditions $G_i \subset M$ to location i. Let $\bar{G}_i$ be the closure of G_i. The vector field $D(i)$ restricted to $\bar{G}_i$ is denoted X_i. The set of edges $E \subset I \times I$ of H is determined by K. $e = (i, i') \in E$ if $\bar{G}_i \cap \bar{G}_{i'} \neq \emptyset$. The enabling condition of edge e is $\bar{G}_i \cap \bar{G}_{i'} \subset K$.

The family $\{X_i\}$ forms a **piecewise C^r vector field** X on M, $1 \leq r \leq \infty$ which captures the dynamics of H. The X_i are referred to as **branches** of X. This piecewise smooth vector field has some kind of orbit portrait, and structural stability of it should mean that perturbing it leaves the orbit portrait unchanged topologically. We proceed to spell this out, thus obtaining a characterization of structural stability of Filippov automata.

A point $q \in K$ is of one of the following four types.

(a) q is a vertex of K.

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(b) q is a **tangency point**: it is not a vertex and at least one of the two branches of X is tangent to K at q. (This includes the possibility that a branch vanishes at q.)

(c) q is a **crossing point**: it is not a vertex, the two branches of X are transverse to K at q, and both point to the same side of K.

(d) q is an **opposition point**: it is not a vertex and the two branches of X oppose each other in the sense that they are transverse to K at q but they point to opposite sides of K.

At an opposition point there is a unique strictly convex combination

$$X^*(q) = \lambda X_i(q) + (1-\lambda)X_j(q)$$

tangent to K at q [1].

Definition 2.1. X^* is the **sliding field**. If $X^*(q) \neq 0$, q is a **sliding point**, while if $X^*(q) = 0$, it is a **singular equilibrium**.

X^* indicates the direction a point should slide along K. See Figure 2. The sliding field is defined on a relatively

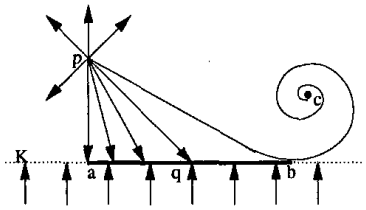

Figure 2: Points slide along K from a to b under the influence of X^*.

open subset of K, which includes neither the vertices of K, nor the points of K at which a branch of X is tangent to K, for at a tangency point, the branches of X do not oppose each other – transversality fails there.

Definition 2.2. A **singularity** of X is a singular equilibrium, a tangent point, or a vertex of K.

Definition 2.3. A **regular orbit** of X is a piecewise smooth curve $\gamma \subset M$ such that $\gamma \cap G_i$ is a trajectory of X_i, $\gamma \cap K$ consists of crossing points, and γ is maximal with respect to these two conditions. A **singular orbit** of X is a smooth curve $\gamma \subset K$ such that γ is either an orbit of X^* or a singularity.

Evidently M decomposes into the disjoint union of orbits, each being regular or singular. They form the **phase portrait** of X. The only periodic or recurrent orbits on K are equilibria. In Figure 2 there is one X-orbit through q (namely (a,b), a regular X-orbit from b to the focus c, and a regular X-orbit from the regular source p to q. The singular point $\{b\}$ is an orbit.

Denote the set of all piecewise C^r vector fields on M by $\mathcal{X}^r_K(M)$. It is a Banach space with respect to the norm $\|X\|_{C^r} = \max_i \|X_i\|_{C^r}$.

Definition 2.4. An **orbit equivalence** is a homeomorphism $h : M \to M$ that sends X-orbits to X'-orbits where $X, X' \in \mathcal{X}^r_K(M)$. The orbit equivalence must preserve the sense (i.e., the direction) of the orbits and send K to itself. If X has a neighborhood $\mathcal{U} \subset \mathcal{X}$ such that each $X' \in \mathcal{U}$ is orbit equivalent to X then X is **structurally stable**.

Definition 2.5. An orbit $\gamma(t)$ **departs** from $q \in K$ if $\lim_{t\to 0+} \gamma(t) = q$. Arrival at q is defined analogously.

Definition 2.6. An **unstable separatrix** is a regular orbit such that either (a) its α-limit set is a regular saddle point, or (b) it departs from a singularity of X. A **stable separatrix** is defined analogously. If a separatrix is simultaneously stable and unstable it is a **separatrix connection**. If unstable separatrices arrive at the same point of an X^*-orbit then they are **related**. Symmetrically, if stable separatrices depart from the same point of K they are related.

3 Generic Filippov Automata

Proposition 3.1. *The branches of the generic $X \in \mathcal{X}$ have the following properties:*

(a) They are Morse-Smale.

(b) None of them vanishes at a point of K.

(c) They are tangent to K at only finitely many points, none of which is a vertex of K, and distinct branches are tangent to K at distinct points.

(d) They are non-colinear except at a finite number of points, none of which is a vertex.

(e) Properties (a)-(d) are stable (i.e., robust) under small perturbations of X.

To prove Proposition 3.1 and to analyze generic singularities we introduce a smooth coordinate chart ϕ in M along an edge $E \subset K$ such that $\phi(E)$ is an interval on the x-axis, say $\phi(E) = [-1,1]$. In this coordinate system the branches X_i, X_j are expressed as vector fields defined on the closed upper half plane and closed lower half plane. On the x-axis we have

$$\begin{aligned} X_i(x,0) &= f_i(x)\Big(\frac{\partial}{\partial x}\Big) + g_i(x)\Big(\frac{\partial}{\partial y}\Big) \\ X_j(x,0) &= f_j(x)\Big(\frac{\partial}{\partial x}\Big) + g_j(x)\Big(\frac{\partial}{\partial y}\Big) \end{aligned} \qquad (1)$$

Definition 3.1. A function $f : [a, b] \to \mathbb{R}$ has **generic zeros** if

(a) $f(a) \neq 0 \neq f(b)$, and

(b) $f(x) = 0$ implies $f'(x) \neq 0$.

Lemma 3.2. *The generic C^r function $f : [-1, 1] \to \mathbb{R}$ has generic zeros.*

Proof: This is a special case of the Thom Transversality Theorem. For (b) is equivalent to 0 being a regular value of f. ∎

Proof: [Proof of Proposition 3.1] Peixoto's Genericity Theorem applied on the surface with smoothly cornered boundary $\bar{G}_i$ states that the generic X_i is Morse-Smale, which is assertion (a) of Proposition 3.1.

As above, introduce a smooth coordinate chart ϕ in which an edge of K is $[-1, 1]$ on the x-axis. Express the branches of X as in (1).

Elaboration of Lemma 3.2 shows that the generic pair of C^r functions $[-1, 1] \to \mathbb{R}$ has no common zeros. This implies that the generic X has no zeros on K, which is assertion (b) of Proposition 3.1.

For the generic X, $g_i(x) = 0$ at only a finite number of points, all of them different from ± 1, and at these zeros, $g_i'(x) \neq 0$. Further, g_i and g_j have no common zeros. This means that generically X_i and X_j are tangent to K at only finitely many points, none of them vertices, and never are they tangent to K at a common point, which is assertion (c) of Proposition 3.1.

Consider any $x_0 \in [-1, 1]$. By (c), either $g_i(x_0) \neq 0$ or $g_j(x_0) \neq 0$, say it is the latter. There is an interval $I \subset [-1, 1]$ containing x_0 on which $g_j \neq 0$. That is, X_j is not tangent to K there. Using a flowbox chart for X_j at I, we may assume that $f_j(x) = 0$ and $g_j(x) > 0$ for all $x \in I$. (In the flowbox coordinates, X_j points straight upwards.) Then colinearity of X_i and X_j occurs when $f_i(x) = 0$. By Lemma 3.2, for the generic X this happens only finitely often, never at an endpoint of I, which is assertion (d) of Proposition 3.1 on the subinterval I. Compactness of $[-1, 1]$ completes the proof of (d).

Assertion (e), stability of (a) - (d) under small perturbations of X, follows from openness of Morse-Smale systems and openness of transversality. ∎

4 Singular Equilibria

The sliding field X^* is defined at non-vertex points $q \in K$ where X_i, X_j oppose each other. These opposition points form a relatively open set in K. With respect to the smooth chart ϕ as in Section 3 and the expression for X_i, X_j along K in (1), we see that g_i and g_j are non-zero and have opposite signs. Since the sliding field is the unique strictly convex combination

$$X^* = \lambda X_i + (1 - \lambda) X_j$$

tangent to K, its vertical component $\lambda g_i(x) + (1 - \lambda) g_j(x)$ is zero. This gives

$$\lambda = \frac{g_j(x)}{g_j(x) - g_i(x)}. \tag{2}$$

Note that this denominator is never zero at opposition points.

Proposition 4.1. *The sliding field X^* is of class C^r. For the generic X, the zeros of X^* are hyperbolic sources or sinks along K.*

Proposition 4.2. *For the generic X, a singularity is either*

- *a singular saddle point,*
- *a singular sink node, or*
- *a singular source node,*
- *a singular saddle node or*
- *a singular grain,*
- *a vertex of K.*

(See the proof for the definitions.) The singularities are finite in number and stable (robust) under small perturbations of X.

Proof: Let $q \in K$ be a singular equilibrium or tangent point of X. By Proposition 3.1 genericity implies that at least one branch of X is transverse to K at q. Say it is X_j. In the flowbox chart ψ as above, $f_j \equiv 0$, $g_j = 1$, and we may take $q = (0, 0)$.

We write a matrix to describe the singularity at q as

$$S = \begin{bmatrix} f_i(0) & g_i(0) \\ f_i'(0) & g_i'(0) \end{bmatrix}.$$

Observe that $g_i(0) \leq 0$. For if $g_i(0)$ is positive then X_i and X_j both point upwards across the x-axis at q, and q is regular, not singular. Genericity implies that the first row of S contains exactly one 0, and neither column is a zero column. Using the symbols $-$, 0, $+$, $\pm$, $*$ to denote an entry that is negative, zero, positive, non-zero, or one whose sign is irrelevant, we get four topologically distinct cases for S. See Figure 3.

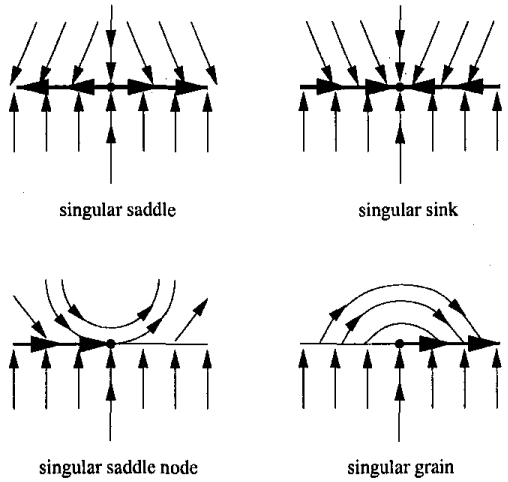

Figure 3: The four generic non-vertex singularities. The heavy lines are X^*-orbits.

Case 1. $S = \begin{bmatrix} 0 & - \\ + & * \end{bmatrix}$. The fact that $f_i(0) = 0$ implies that X_i, X_j are colinear at q and there is a unique X_i-orbit that arrives at the origin from above. Since $f_i'(0)$ is positive this gives a singular saddle with regular stable separatrices.

Case 2. $\begin{bmatrix} 0 & - \\ - & * \end{bmatrix}$. The fact that $f_i(0) = 0$ implies that X_i, X_j are colinear at q and there is a unique X_i-orbit that arrives at the origin from above. Since $f_i'(0)$ is negative this gives a singular sink with regular stable separatrices. See Figure 3.

Case 3. $\begin{bmatrix} \pm & 0 \\ * & + \end{bmatrix}$. Since $g_i(0) = 0$ and $g_i'(0)$ is positive, we get a singular saddle node with three regular separatrices, two stable and one unstable. The sign of $f_i(0)$ only affects whether the arrows point right or left.

Case 4. $\begin{bmatrix} \pm & 0 \\ * & - \end{bmatrix}$. Since $g_i(0) = 0$ and $g_i'(0)$ is negative, we get a singular grain with one regular stable separatrix. The sign of $f_i(0)$ only affects whether the arrows point right or left.

Four homeomorphic pictures, but with the orientations of all orbits reversed, are achieved by starting with the field X_j pointing straight down instead of straight up. Finiteness of the singular equilibria and tangent points is a consequence of (c), (d) in Proposition 3.1. Stability under perturbation of X follows from openness of transversality. ∎

5 Singular Sectors

Definition 5.1. Let v be a vertex of K. A **base orbit** of X at v is an orbit, singular or regular, that arrives at v or departs from v, but is not v itself.

Proposition 5.1. *The generic X has at least one base orbit at each vertex. In fact, one of the following two possibilities occurs.*

*(a) There exists no singular base orbit, in which case X has a **singular focus** at v: all orbits near v are regular and either all arrive at v or all depart from v.*

(b) There exists a singular base orbit, in which case there exist at most $2n$ base orbits at v, where n is the number of edges of K at v.

Proof: Because K locally separates M, $n \geq 2$. We draw a circle C around v and refer to the component of G_i inside C as the **corner** V_i of G_i, $i = 1, \ldots, n$. (The term "sector" applies equally well to V_i, but we refer also to the dynamical sectors at v.) The angle between the edges of V_i at v is the **aperture** of V_i. We choose C small – it encloses no singular equilibria or tangent points, and the edges of K at v are arcs from C to v.

Suppose that the aperture of some corner V_i is $\geq \pi$. Generically $X_i(v)$ is non-zero and is parallel to neither edge of V_i. From now on we assume that the aperture of all corners is $< \pi$. In particular, $n \geq 3$.

Proposition 3.1 states that the generic X is not tangent to an edge of K at v. Thus, when the circle C is small and $1 \leq i \leq n$, either (c) X_i points inward across both edges of V_i, or points outward across both edges, or (d) X_i points inward across one edge of V_i and outward across the other. Each corner contains at most one base orbit in its interior.

Assume that there is no singular base orbit at v. Then X_i never opposes X_{i+1} across the common edge $V_i \cap V_{i+1}$, and we get an arc α of a regular X-orbit that starts at $a \in E_1$, an edge of V_1 and continues through each of the corners at v until it comes back to E_1, say at the point a'.

The vectors $X_i(v)$ are fixed, and so are the apertures of the corners V_i. The arc α is an amalgam of nearly straight segments, $\alpha_1, \ldots, \alpha_n$. The exterior angle θ_i between α_i and α_{i+1} tends to a definite non-zero limit as the circle C shrinks to v. Generically $\Theta = \sum_{i=1}^n \theta_i \neq 2\pi$, and thus the points at which α crosses E_1 are distinct. If $\Theta > 2\pi$ then a' lies closer than a to v along E_1. In fact, for a constant $c < 1$, $|a' - v| \leq c|a - v|$. The same constant c works for all a sufficiently close to v. See Figure 4. The length of

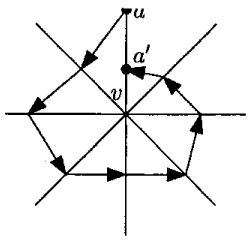

Figure 4: A singular focus.

time it takes for α to make the circuit through the corners at v is proportional to $|a - v|$. For $X_i(z)$ tends to the non-zero vector $X_i(v)$ as $z \to v$ in V_i. Thus the X-orbit through a arrives at v (in finite time), and is a base orbit. In fact the local orbit portrait at v is that of a focus where all orbits arrive at v.

If $\Theta < 2\pi$ it is the opposite. All orbits near v depart from v. This completes the proof of assertion (a), lack of a singular base orbit implies a singular focus.

Now assume that there is at least one singular base orbit β. It is an edge of adjacent corners V_i, V_{i+1} such that X_i and X_{i+1} oppose each other across β. No base orbit can spiral around v because it is blocked by β. Thus, the only possible base orbits are the n edges of K at v (they would be singular base orbits) plus n regular base orbits, one interior to each corner. This completes the proof of assertion (b), the existence of one singular base orbit implies there are at most $2n$ base orbits at v. ∎

Definition 5.2. A **singular sector** is a region S bounded by an arc of the small circle C at v together with two consecutive base orbits. S must contain no other base orbits.

Corollary 5.2. *The orbit portrait for the generic X inside a singular sector is either singular hyperbolic, singular parabolic, or singular elliptic. See Figure 5.*

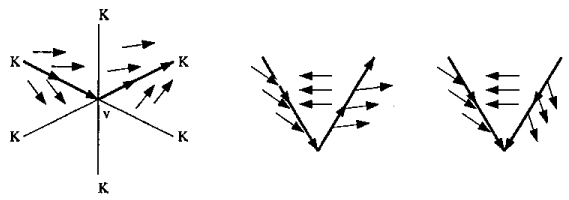

Figure 5: Singular sectors: hyperbolic, elliptic, and parabolic.

6 Separatrix Connections

In this section we examine various forms of separatrix connections, and in particular we show how to bury some of them.

Examples of separatrices are shown in Figure 3. Note that a singular sink q has stable separatrices – the unique *regular* orbits that arrive at q. Likewise a singular source has unstable separatrices. A base orbit is also a separatrix. It is easy to see that structural stability fails if there are separatrix connections or separatrix relations. Also, singular orbits never connect singular saddles, so there is no need to exclude them by a genericity argument. See Proposition 6.3.

Proposition 6.1. *For the generic X, no separatrix connections or relations occur in the neighborhood of K. Moreover there are restrictions on how the equilibria appear along K.*

Definition 6.1. An unstable separatrix is **tame** if its ω-limit set is a regular point sink, a regular periodic orbit sink, or if it arrives at a sliding point. Time reversal gives the corresponding definition for stable separatrices.

Proposition 6.2. *If a separatrix is tame then it stays tame under small perturbations of X.*

Proposition 6.3. *A separatrix connection between a singularity and a singularity or regular saddle point can be tamed by a small perturbation of X. Also, separatrix relations can be broken by small perturbations of X, and once broken, they stay broken under subsequent sufficiently small perturbations of X.*

Corollary 6.4. *The generic X has no separatrix connection between a singularity and a singularity or regular saddle point. This remains true for small perturbations of X.*

7 Recurrence and Periodicity

In this section we prove that if you have recurrence then you can connect separatrices.

Proposition 7.1. *Assume that the X-orbit through p is non-trivially recurrent. Then it is regular and through p there passes a regular, smooth Jordan curve J everywhere transverse to X.*

Proof: Singular orbits are points and sliding curves. They are not non-trivially recurrent. The existence of J is proved in the same way as for flows. Although the orbit arc from p to a closest return may have a few corners where it crosses K, the construction is unaffected. ∎

The first return map, or Poincare map P, is naturally defined by the X-orbits that leave J at y and return at $P(y)$. Transversality implies that the domain of definition of P is an open subset $D \subset J$. It consists of open intervals I or it equals J. The X-orbits through I are regular, and we get an open-sided strip from I to $P(I)$. Let a be an endpoint of I. Its orbit does not return to J but it does stay on the boundary of the strip. Thus the forward orbit of a ends at a point a'. For the generic X, a' can be a regular saddle point, a singular saddle node, or a vertex. Thus a lies on a separatrix that does not return to J after leaving a. There are only finitely many separatrices and therefore D is either J or a finite union of intervals I whose endpoints lie on stable separatrices that leave J and go directly to regular saddles, singular saddle nodes, or vertices.

Proposition 7.2. *Assume that the Poincaré map is defined on the whole closed transversal J and some points of J are non-trivially recurrent. Then M is the torus, all points are regular non-equilibria, and a small perturbation of X produces a periodic orbit.*

Proposition 7.3. *Assume that the Poincaré map is defined on $D \neq J$ and a point $p \in D$ is non-trivially recurrent. Then there exists a regular unstable separatrix that accumulates at p.*

Proposition 7.4. *As above, a small perturbation exists that leaves all separatrix connections intact and produces one more.*

8 Main Results

Proposition 8.1. *If X has no recurrence then the ω-limit set of an orbit of X is either empty (the orbit arrives at a singularity or sliding point in finite time), or is a regular equilibrium, or is a vertex, or is a finite* ***graphic cycle*** *that consists of separatrix connections.*

Proposition 8.2. *A graphic cycle that is the ω-limit of an unstable separatrix σ can be perturbed to produce a regular period orbit that tames σ.*

The follow two theorems are our main results on structural stability of Filippov automata.

Theorem 8.3. *The generic $X \in \mathcal{X}$ is structurally stable.*

Theorem 8.4. *The following conditions characterize structural stability of $X \in \mathcal{X}$.*

- *The conditions listed in Proposition 3.1.*
- *Hyperbolicity of all periodic orbits.*
- *No separatrix connections or relations.*
- *Only trivially recurrent orbits.*

The proof imitates the Palis and deMelo approach to Peixoto's structural stability result. That is, if you have recurrence, you can keep connecting separatrices until there are no more left to connect, and then you can tame one more separatrix. Then apply induction. A special case occurs when there are no separatrices, namely, the torus with recurrence.

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